Math 3450 - Homework # 4
Functions

Part 1 - Composition of functions

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1. Let $f : \mathbb{Z} \to \mathbb{Z}$ and $g : \mathbb{Z} \to \mathbb{Z}$ where $f(x) = x^2$ and $g(x) = 2x + 1$.

   (a) Compute $(f \circ g)(2)$ and $(g \circ f)(-1)$.

   (b) Give formulas for $(f \circ g)(x)$ and $(g \circ f)(x)$.

2. Let $f : \mathbb{Z} \times \mathbb{Z} \to Z$ be given by $f(m, n) = m + n$ and $g : \mathbb{Z} \to Z \times \mathbb{Z}$ be given by $g(x) = (x, x)$.

   (a) Compute $(g \circ f)(1, 2)$ and $(f \circ g)(-1)$

   (b) Find formulas for $(g \circ f)(m, n)$ and $(f \circ g)(x)$.

3. Let $f : \mathbb{Z} \times \mathbb{Z} \to Z \times \mathbb{Z}$ be given by $f(m, n) = (3m - 4n, 2m + n)$ and $g : \mathbb{Z} \times \mathbb{Z} \to Z \times \mathbb{Z}$ be given by $g(m, n) = (5m + n, m)$.

   (a) Compute $(g \circ f)(-1, 1)$ and $(f \circ g)(2, 3)$

   (b) Find formulas for $(g \circ f)(m, n)$ and $(f \circ g)(m, n)$.

4. Let $A = \{1, 2, 3, 4\}$. Let $i_A : A \to A$ be the identity function on A. That is, $i_A(x) = x$ for all $x \in A$.

   (a) Let $f : A \to A$ where $f(1) = 3$, $f(2) = 1$, $f(3) = 2$, and $f(4) = 4$. Draw a picture of f. Draw a picture of f^{-1}. Show that $f \circ f^{-1} = i_A$ and $f^{-1} \circ f = i_A$.

   (b) Let $g : A \to A$ where $g(1) = 1$, $g(2) = 3$, $g(3) = 4$, and $g(4) = 2$. Draw a picture of g. Draw a picture of g^{-1}. Show that $g \circ g^{-1} = i_A$ and $g^{-1} \circ g = i_A$.

5. Give an example of $f : A \to B$ and $g : B \to C$ where the following are true:

   (a) f is not onto, but $g \circ f$ is onto.

   (b) g is not one-to-one, but $g \circ f$ is one-to-one.

Part 2 - Well-defined functions

6. Let $f : \mathbb{Q} \to \mathbb{Z}$ be defined by $f(m/n) = m$. For example, $f(2/9) = 2$ and $f(5/10) = 5$. Show that f is not a well-defined function.

Part 3 - Applying functions to sets

7. Let $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = x^2 - 2$.
   (a) Compute $f(\{1, 3, -1, 2\})$
   (b) Compute $f([0, 1])$ where $[0, 1] = \{x \mid 0 \leq x \leq 1\}$.
   (c) Compute $f^{-1}(\{-4, -3, -2, 1, 6\}$
   (d) Compute $f^{-1}([0, 1))$ where $[0, 1) = \{y \mid 0 \leq y < 1\}$.
   (e) Compute $f^{-1}([-3, -1))$ where $[-3, -1) = \{y \mid -3 \leq y < -1\}$.

8. Let $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = \begin{cases} x - 1 & \text{if } x \leq 0 \\ x^2 & \text{if } x \leq 0 \end{cases}$
   (a) Compute $f(\{-10, 0, 5, 7\})$
   (b) Compute $f([-1, 2])$ where $[-1, 2] = \{x \mid -1 \leq x \leq 2\}$.
   (c) Compute $f^{-1}(\{-2, -1, 1, 6\}$
   (d) Compute $f^{-1}([-3, 2])$ $[-3, 2] = \{y \mid -3 \leq y \leq 2\}$

9. Let $f : \mathbb{Z} \times \mathbb{Z} \to Z$ be given by $f(m, n) = m + n$ and $g : \mathbb{Z} \to Z \times \mathbb{Z}$ be given by $g(x) = (x, x)$.
   (a) Calculate $g^{-1}(A)$ where $A = \{(0, 0), (1, 1), (1, 2)\}$
   (b) Calculate $f^{-1}(B)$ where $B = \{0\}$

10. Let $f : \mathbb{Z} \times \mathbb{Z} \to Z \times \mathbb{Z}$ be given by $f(m, n) = (3m - 4n, 2m + n)$ and $g : \mathbb{Z} \times \mathbb{Z} \to Z \times \mathbb{Z}$ be given by $g(m, n) = (5m + n, m)$.
    (a) Calculate $g^{-1}(A)$ where $A = \{(0, 0), (1, -1)\}$
    (b) Calculate $f^{-1}(B)$ where $B = \{(0, 0), (1, -1)\}$